Midsemestral Examination
B. Math. (Hons.) 2nd year
Group Theory
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September 18, 2023
Attempt only 4 questions; Be Brief.
You may use Sylow's theorem or Orbit-Stabilizer theorem etc. but state precisely what you use.

Q 1.
(i) Show that $(\mathbb{Z}/24\mathbb{Z})^{\times} \cong \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$.
(ii) Prove that the group $Hom(\mathbb{Z}_n, \mathbb{C}^*)$ of homomorphisms from the additive group $\mathbb{Z}_n$ to the multiplicative group $\mathbb{C}^*$ forms a cyclic group isomorphic to $\mathbb{Z}_n$.
Hint. Any element of $Hom(\mathbb{Z}_n, \mathbb{C}^*)$ is determined by the image of 1 which must be an n-th root of unity.

Q 2.
(i) Prove that a group of order 2023 must be abelian.
(ii) Use Sylow's theorems to prove that no group of order 80 can be simple.
Hint. Look at the number of elements of order 5.

OR

(i) Show that any group of order $3p$ where p is a prime of the form $6k+5$ must be cyclic.
(ii) Prove that in a group of order 105, the 5-Sylow subgroup and the 7-Sylow subgroup are both unique.
Hint. Show that there is a subgroup of order 35, and it must be cyclic.

Q 3.
(i) Describe all the conjugacy classes and the center of the dihedral group D_8 of order 8.
(ii) If G ia a p-group, and H is a nontrivial, normal subgroup, prove that $H \cap Z(G)$ is also non-trivial.
Hint. Look at the conjugacy action on H.

OR

(i) If a group of order 33 acts on a set of cardinality 19, prove that there must be a fixed point.
(ii) Let G be a group acting on two sets S, T. Assume that $s \in S, t \in T$ have the property that each $g \in G$ fixes either s or t. Prove that the whole of G fixes either s or t.
Hint. If not, consider $a, b \in G$ such that $as \neq s, bt \neq t$ and consider abs, abt.

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Q 4.
(i) If G is a finite group, H is a subgroup and Q is a p-Sylow subgroup of H, prove that there exists a p-Sylow subgroup P of G such that $P \cap H = Q$.
(ii) If G is a finite abelian group, and p^r is a prime p dividing $O(G)$, prove that $\{x \in G : \text{order of } x \text{ is a power of } p\}$ is the unique p-Sylow subgroup of G.

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Q 5.
(i) Let $T : G \to G$ be an automorphism of the finite group G such that $T(x) = x$ if, and only if, $x = 1$. Prove that each element of G can be written as $x^{-1}T(x)$ for some $x \in G$. Further, if T^r is the identity map, show that for each $g \in G$, we have

$$gT(g)T^2(g)\cdots T^{r-1}(g) = 1.$$

(ii) If the center $Z(G)$ of a group G has finite index in it, prove that $[G, G]$ is finite.

OR

(i) If G is a finite group, and $\Phi(G)$ is the intersection of all maximal subgroups of G, prove that $\Phi(G)$ is the set of 'nongenerators'; that is, whenever $S \cup \{x\}$ generates G with $x \in \Phi(G)$, then S generates G.
(ii) If G is a finite, simple group in which every proper subgroup is abelian. Prove that for any two different maximal subgroups M, N we have $M \cap N = \{1\}$.